

APPLICATIONS AND INTERPRETATIONS (AI)

STANDARD LEVEL (SL)

REVISION

TOPIC 2: FUNCTIONS

2.1

Different forms of the equation of a straight line.

Gradient; intercepts.

Lines with gradients m_1 and m_2

Parallel lines $m_1 = m_2$

Perpendicular lines $m_1 \times m_2 = -1$

$y = mx + c$ (gradient-intercept form).

$ax + by + d = 0$ (general form).

$y - y_1 = m(x - x_1)$ (point-gradient form).

Calculate gradients of inclines such as mountain roads, bridges, etc.

Q1.

Determine the equation of the line passing through the points (2,4) and (5,2). Write your answer in the form $Ax + By + D = 0$, where $A, B, D \in \mathbb{Z}$.

Q2.

Determine the equation of the perpendicular bisector of the line connecting the points (2,5) and (6,7).

Q3.

Line L_1 is tangent to the graph of a function $f(x)$ at the point $P(3, -1)$. Line L_2 is given by the equation $y = -\frac{1}{2}x - \frac{5}{2}$ and is perpendicular to L_1 .

- (a) Write down the gradient of L_1 .
- (b) Find the equation of L_1 in the form $y = mx + c$.

Q4.

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

Year (x)	1708	1758	1808	1858	1908	1958
Year °C (y)	8.73	9.22	9.10	9.12	9.13	9.45

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates (1708, 8.73) and (1958, 9.45).

- (a) Calculate the gradient of the straight line that passes through these two points.
- (b.i) Interpret the meaning of the gradient in the context of the question.
- (b.ii) State appropriate units for the gradient.
- (c) Find the equation of this line giving your answer in the form $y = mx + c$.
- (d) Use Tami's model to estimate the mean annual temperature in the year 2000.

2.2

Concept of a function, domain, range and graph.

Function notation, for example $f(x)$, $v(t)$, $C(n)$.

The concept of a function as a mathematical model

Example: $f(x) = \sqrt{2-x}$ the domain is $x \leq 2$, range is $f(x) \geq 0$.

A graph is helpful in visualizing the range.

Informal concept that an inverse function reverses or undoes the effect of a function.

Inverse function as a reflection in the line $y = x$, and the notation $f^{-1}(x)$.

Example: Solving $f(x) = 10$ is equivalent to finding $f^{-1}(10)$.

Students should be aware that inverse functions exist for one to one functions; the domain of $f^{-1}(x)$ is equal to the range of $f(x)$.

Q5.

Let $f(x) = 2 + \frac{3}{1+x}$.

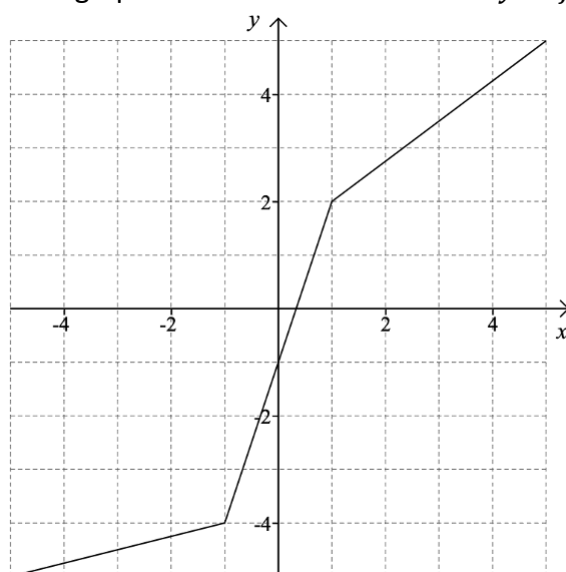
(a) For $f(x)$ find

(i) the domain

(ii) the range

(b) Find $f^{-1}(1)$.

Q6. The graph below shows the function $y = f(x)$.



Draw the graph of the function $y = f^{-1}(x)$ on the same axes.

Q7.

Consider the following function:

$$h(x) = \frac{2}{\sqrt{x-1}} + \frac{1}{2}, \text{ for } x > 1.$$

- (a) Find $h^{-1}(1)$.
- (b) Find the domain of $h^{-1}(x)$.

Q8.

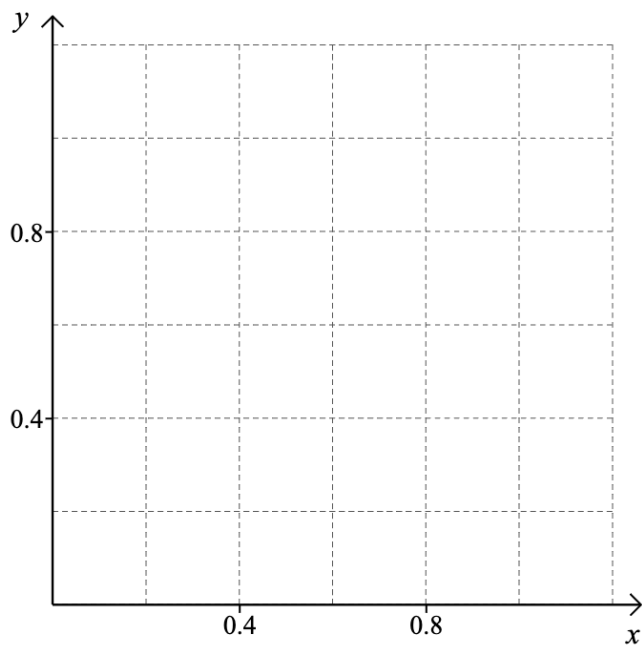
A function is defined by $f(x) = 2 - \frac{12}{x+5}$ for $-7 \leq x \leq 7$, $x \neq -5$.

- (a) Find the range of f .
- (b) Find the value of $f^{-1}(0)$.

2.3	
The graph of a function; its equation $y = f(x)$.	Students should be aware of the difference between the command terms “draw” and “sketch”.
Creating a sketch from information given or a context, including transferring a graph from screen to paper. Using technology to graph functions including their sums and differences.	All axes and key features should be labelled. This may include functions not specifically mentioned in topic 2.

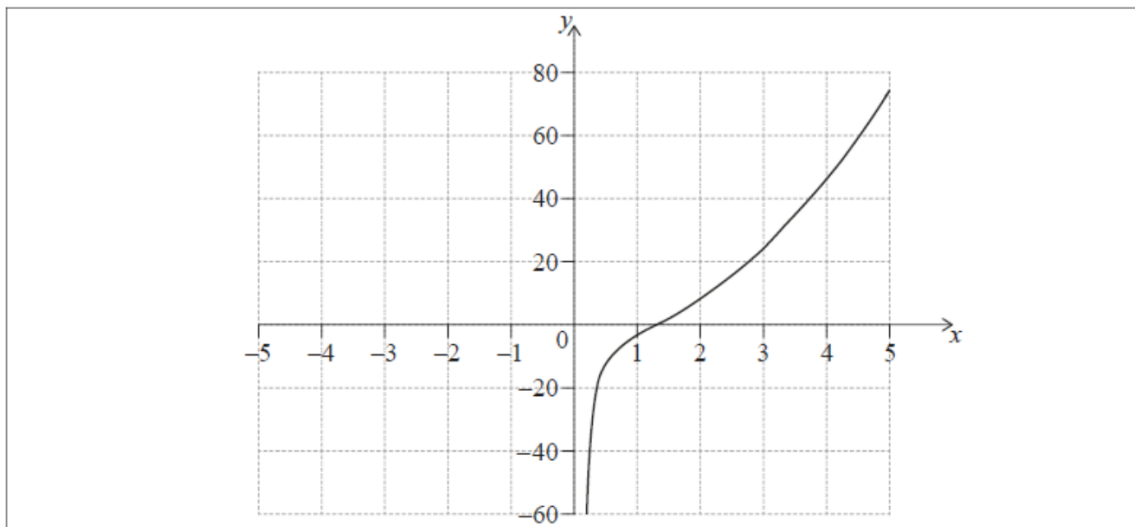
Q9. Let $f(x) = \sqrt{1-x}$ where $x \in [0,1]$.

Sketch the graph of $y = f(x)$ on the axes below.



Q10.

Consider $f(x) = 3x^2 - \frac{5}{x}$, $x \neq 0$. The graph of f for $0 < x \leq 5$ is shown on the following axes.



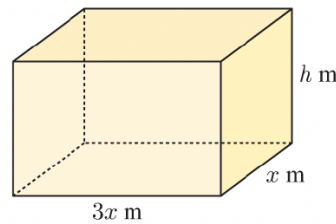
- (a.i) Sketch the graph of f , for $-5 \leq x < 0$, on the same axes.
- (a.ii) Write down the x -coordinate of the local minimum point.
- (b) Use your graphic display calculator to find the solutions to the equation $f(x) = 20$
- (c) Write down the equation of the vertical asymptote for the graph of f .

2.4	
Determine key features of graphs.	Maximum and minimum values; intercepts; symmetry; vertex; zeros of functions or roots of equations; vertical and horizontal asymptotes using graphing technology.
Finding the point of intersection of two curves or lines using technology.	

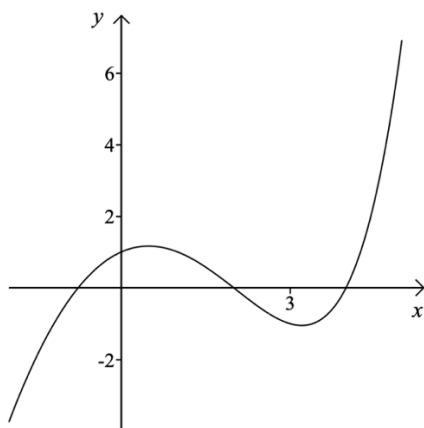
Q11.

A closed box is to be constructed with length three times its width. The volume of the box must be 6 m^3 .

- a** If the box is x metres wide and h metres high, show that $h = \frac{2}{x^2}$.
- b** Hence show that the surface area of the box is given by $A(x) = 6x^2 + \frac{16}{x} \text{ m}^2$.
- c** Find the vertical asymptote of $y = A(x)$.
- d** Draw the graph of $y = A(x)$.
- e** Use your graph to estimate the surface area of a box with width 2 metres. Check your answer using technology.
- f** Find the coordinates of the turning point of $y = A(x)$. Explain the significance of this point.



Q12. The diagram below shows the graph of $y = 2^x - x^2$.



Find the coordinates of:

- The axes intercepts
- The turning points

Q13.

Consider the function $f(x) = \frac{27}{x^2} - 16x$, $x \neq 0$.

- (a) Sketch the graph of $y = f(x)$, for $-4 \leq x \leq 3$ and $-50 \leq y \leq 100$.
- (b.i) Use your graphic display calculator to find the zero of $f(x)$.
- (b.ii) Use your graphic display calculator to find the coordinates of the local minimum point.

Q14. The graphs $y = -x^2 + 2$ and $y = e^x$ intersect at two points.

- a) Sketch the graphs on the same axes showing all axes intercepts.
- b) Find the coordinates of the points of intersection.

2.5	
Modelling with the following functions:	
Linear models. $f(x) = mx + c.$	<i>Including piecewise linear models, for example horizontal distances of an object to a wall, depth of a swimming pool, mobile phone charges.</i>
Quadratic models. $f(x) = ax^2 + bx + c; a \neq 0.$ Axis of symmetry, vertex, zeros and roots, intercepts on the x-axis and y-axis.	<i>Technology can be used to find roots.</i>
Exponential growth and decay models. $f(x) = ka^x + c$ $f(x) = ka^{-x} + c, (for a > 0)$ $f(x) = ke^{rx} + c$ Equation of a horizontal asymptote.	
Direct/inverse variation: $f(x) = ax^n, n \in \mathbb{Z}$ The y-axis as a vertical asymptote when $n < 0.$	
Cubic models: $f(x) = ax^3 + bx^2 + cx + d$	
Sinusoidal models: $f(x) = asin(bx) + d,$ $f(x) = acos(bx) + d$	<i>Students will not be expected to translate between $\sin x$ and $\cos x$, and will only be required to predict or find amplitude (a), period $\left(\frac{360^\circ}{b}\right)$, or equation of the principal axis ($y = d$).</i>

Miss Zahkna

Olava's Pizza Company supplies and delivers large cheese pizzas.

The total cost to the customer, C , in Papua New Guinean Kina (PGK), is modelled by the function

$$C(n) = 34.50n + 8.50, \quad n \geq 2, \quad n \in \mathbb{Z},$$

where n , is the number of large cheese pizzas ordered. This total cost includes a fixed cost for delivery.

- (a) State, in the context of the question,
- (i) what the value of 34.50 represents;
 - (ii) what the value of 8.50 represents. [2]
- (b) Write down the minimum number of pizzas that can be ordered. [1]
- Kaelani has 450 PGK.
- (c) Find the maximum number of large cheese pizzas that Kaelani can order from Olava's Pizza Company. [3]

Q15.

Professor Vinculum investigated the migration season of the Bulbul bird from their natural wetlands to a warmer climate.

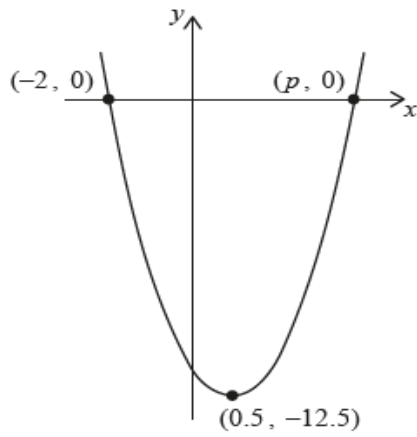
He found that during the migration season their population, P could be modelled by $P = 1350 + 400(1.25)^{-t}, t \geq 0$, where t is the number of days since the start of the migration season.

- (a.i) Find the population of the Bulbul birds at the start of the migration season. [1]
- (a.ii) Find the population of the Bulbul birds after 5 days.
- (b) Calculate the time taken for the population to decrease below 1400.
- (c) According to this model, find the smallest possible population of Bulbul birds during the migration season.

Q16.

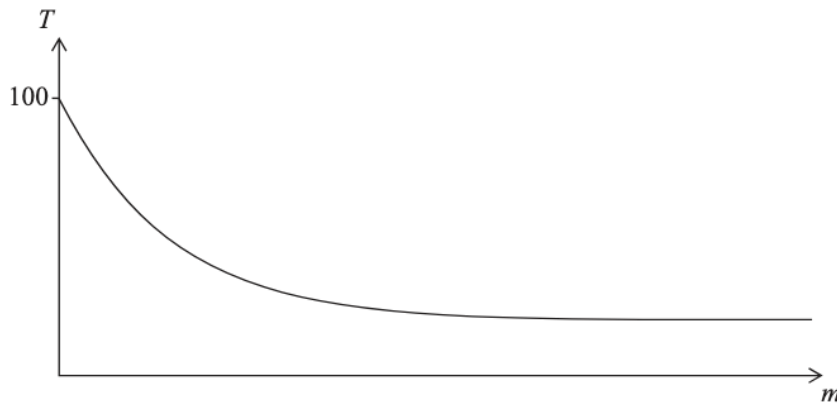
Consider the function $f(x) = ax^2 + bx + c$. The graph of $y = f(x)$ is shown in the diagram. The vertex of the graph has coordinates $(0.5, -12.5)$. The graph intersects the x -axis at two points, $(-2, 0)$ and $(p, 0)$.

diagram not to scale



- (a) Find the value of p .
- (b) Find the value of
- a .
 - b .
 - c .
- (c) Write down the equation of the axis of symmetry of the graph.

A cup of boiling water is placed in a room to cool. The temperature of the room is 20°C . This situation can be modelled by the exponential function $T = a + b(k^{-m})$, where T is the temperature of the water, in $^{\circ}\text{C}$, and m is the number of minutes for which the cup has been placed in the room. A sketch of the situation is given as follows.



- (a) Explain why $a = 20$. [2]

Initially, at $m = 0$, the temperature of the water is 100°C .

- (b) Find the value of b . [2]

After being placed in the room for one minute, the temperature of the water is 84°C .

- (c) Show that $k = 1.25$. [2]

- (d) Find the temperature of the water three minutes after it has been placed in the room. [2]

- (e) Find the total time needed for the water to reach a temperature of 35°C . Give your answer in minutes and seconds, correct to the nearest second. [3]

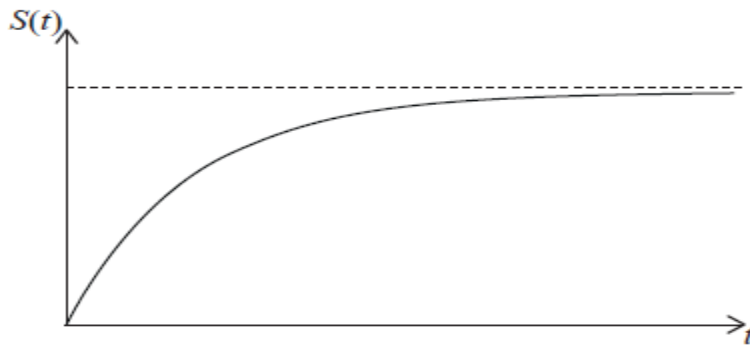
Q17. Jean-Pierre jumps out of an airplane that is flying at constant altitude. Before opening his parachute, he goes through a period of freefall.

Jean-Pierre's vertical speed during the time of freefall, S , in m s^{-1} , is modelled by the following function.

$$S(t) = K - 60(1.2^{-t}), \quad t \geq 0$$

where t , is the number of seconds after he jumps out of the airplane, and K is a constant. A sketch of Jean-Pierre's vertical speed against time is shown below.

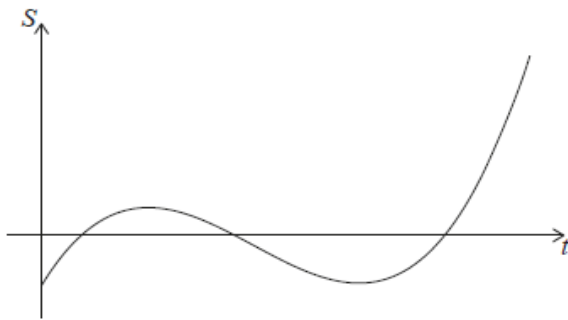
Miss Zahkna



Jean-Pierre's initial vertical speed is 0 m s^{-1} .

- (a) Find the value of K .
- (b) In the context of the model, state what the horizontal asymptote represents.
- (c) Find Jean-Pierre's vertical speed after 10 seconds. Give your answer in km h^{-1} .

Q18. The graph below shows the average savings, S thousand dollars, of a group of university graduates as a function of t , the number of years after graduating from university.



- (a) Write down one feature of this graph which suggests a cubic function might be appropriate to model this scenario.

The equation of the model can be expressed in the form $S = at^3 + bt^2 + ct + d$, where a , b , c and d are real constants.

The graph of the model must pass through the following four points.

t	0	1	2	3
S	-5	3	-1	-5

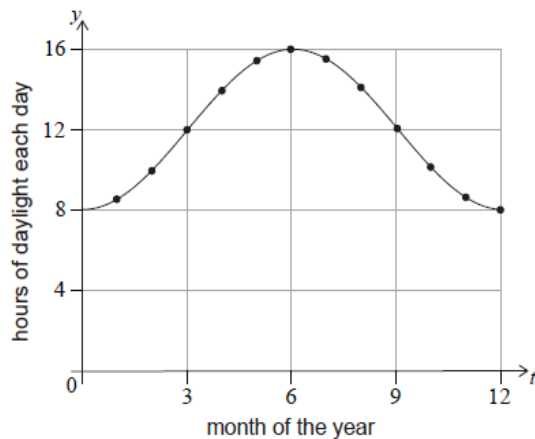
- (b.i) Write down the value of d .
- (b.ii) Write down three simultaneous equations for a , b and c .
- (b.iii) Hence, or otherwise, find the values of a , b and c .

A negative value of S indicates that a graduate is expected to be in debt.

- (c) Use the model to determine the total length of time, in years, for which a graduate is expected to be in debt after graduating from university.

Q19. Boris recorded the number of daylight hours on the first day of each month in a northern hemisphere town.

This data was plotted onto a scatter diagram. The points were then joined by a smooth curve, with minimum point $(0, 8)$ and maximum point $(6, 16)$ as shown in the following diagram.



Let the curve in the diagram be $y = f(t)$, where t is the time, measured in months, since Boris first recorded these values.

Boris thinks that $f(t)$ might be modelled by a quadratic function.

(a) Write down one reason why a quadratic function would not be a good model for the number of hours of daylight per day, across a number of years.

Paula thinks that a better model is $f(t) = a \cos(bt) + d$, $t \geq 0$, for specific values of a , b and d .

For Paula's model, use the diagram to write down

(b.i) the amplitude.

(b.ii) the period.

(b.iii) the equation of the principal axis.

(c) Hence or otherwise find the equation of this model in the form:

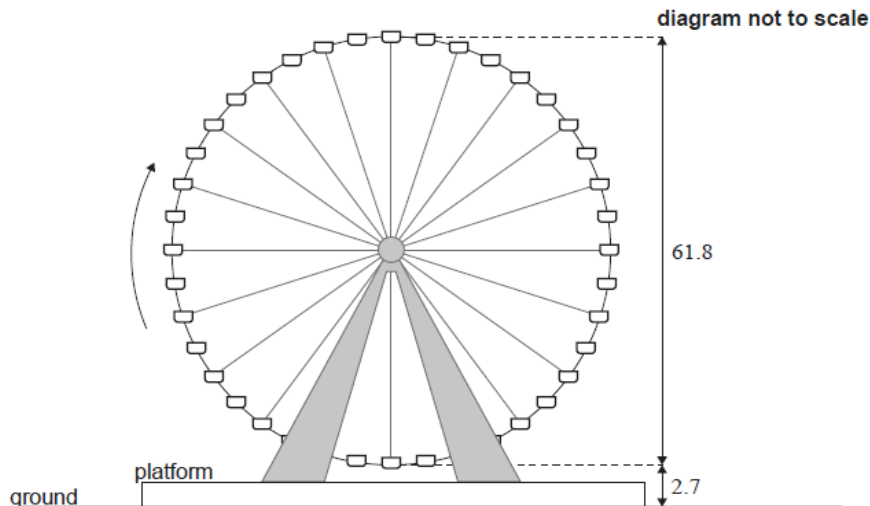
$$f(t) = a \cos(bt) + d$$

(d) For the first year of the model, find the length of time when there are more than 10 hours and 30 minutes of daylight per day.

The true maximum number of daylight hours was 16 hours and 14 minutes.

(e) Calculate the percentage error in the maximum number of daylight hours Boris recorded in the diagram.

Q20. The Texas Star is a Ferris wheel at the state fair in Dallas. The Ferris wheel has a diameter of 61.8 m. To begin the ride, a passenger gets into a chair at the lowest point on the wheel, which is 2.7 m above the ground, as shown in the following diagram. A ride consists of multiple revolutions, and the Ferris wheel makes 1.5 revolutions per minute.



The height of a chair above the ground, h , measured in metres, during a ride on the Ferris wheel can be modelled by the function $h(t) = -a \cos(bt) + d$, where t is the time, in seconds, since a passenger began their ride.

Calculate the value of

(a.i) a .

(a.ii) b .

(a.iii) d .

A ride on the Ferris wheel lasts for 12 minutes in total.

(b) Calculate the number of revolutions of the Ferris wheel per ride.

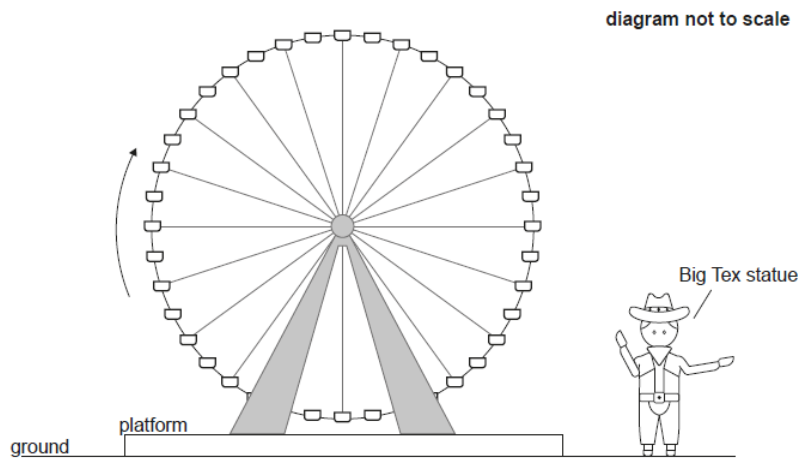
For exactly one ride on the Ferris wheel, suggest

(c.i) an appropriate domain for $h(t)$.

(c.ii) an appropriate range for $h(t)$.

Miss Zahkna

Big Tex is a 16.7 metre-tall cowboy statue that stands on the horizontal ground next to the Ferris wheel.



[Source: Aline Escobar., n.d. Cowboy. [image online] Available at: <https://thenounproject.com/search/?q=cowboy&i=1080130>

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<https://creativecommons.org/licenses/by-sa/3.0/deed.en> [Accessed 13/05/2021]. Source adapted.]

(d) By considering the graph of $h(t)$, determine the length of time during one revolution of the Ferris wheel for which the chair is higher than the cowboy statue.

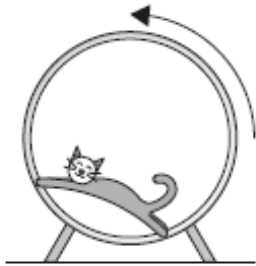
There is a plan to relocate the Texas Star Ferris wheel onto a taller platform which will increase the maximum height of the Ferris wheel to 65.2 m. This will change the value of one parameter, a , b or d , found in part (a).

(e.i) Identify which parameter will change.

(e.ii) Find the new value of the parameter identified in part (e)(i).

Miss Zahkna

Q21. A cat runs inside a circular exercise wheel, making the wheel spin at a constant rate in an anticlockwise direction. The height, h cm, of a fixed point, P, on the wheel can be modelled by $h(t) = a \sin(bt) + c$ where t is the time in seconds and $a, b, c \in \mathbb{R}^+$.



When $t = 0$, point P is at a height of 78 cm.

(a) Write down the value of c .

When $t = 4$, point P first reaches its maximum height of 143 cm.

(b.i) Find the value of a .

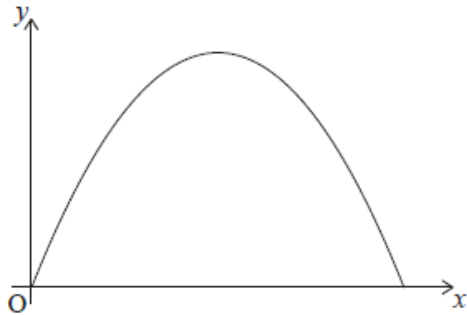
(b.ii) Find the value of b .

(c) Write down the minimum height of point P.

Later, the cat is tired, and it takes twice as long for point P to complete one revolution at a new constant rate.

(d) Write down the new value of b .

Q22. The cross-section of an arched entrance into the ballroom of a hotel is in the shape of a parabola. This cross-section can be modelled by part of the graph $y = -1.6x^2 + 4.48x$, where y is the height of the archway, in metres, at a horizontal distance, x metres, from the point O , in the bottom corner of the archway.



(a) Determine an equation for the axis of symmetry of the parabola that models the archway.

To prepare for an event, a square-based crate that is 1.6 m wide and 2.0 m high is to be moved through the archway into the ballroom. The crate must remain upright while it is being moved.

(b) Determine whether the crate will fit through the archway. Justify your answer.

2.6	
<p>Modelling skills: Use the modelling process described in the “mathematical modelling” section to create, fit and use the theoretical models in section SL2.5 and their graphs.</p>	<p><i>Fitting models using regression is covered in topic 4.</i></p>
<p>Develop and fit the model: Given a context recognize and choose an appropriate model and possible parameters. Determine a reasonable domain for a model.</p>	
<p>Find the parameters of a model.</p>	<p><i>By setting up and solving equations simultaneously (using technology), by consideration of initial conditions or by substitution of points into a given function. At SL, students will not be expected to perform non-linear regressions, but will be expected to set up and solve up to three linear equations in three variables using technology.</i></p>
<p>Test and reflect upon the model: Comment on the appropriateness and reasonableness of a model. Justify the choice of a particular model, based on the shape of the data, properties of the curve and/or on the context of the situation.</p>	
<p>Use the model: Reading, interpreting and making predictions based on the model.</p>	<p><i>Students should be aware of the dangers of extrapolation.</i></p>

Q23.

3000 citizens attended a carnival. Let x be the number of adults attending the carnival and y be the number of children attending the carnival.

- (a) Write down an equation in x and y . [1]

The cost of an adult ticket and a child ticket were set to be USD 18 and USD 8 respectively. The total cost of tickets sold in the carnival was USD 36000.

- (b) Write down another equation in x and y . [1]
- (c) Write down the values of x and y . [2]
- (d) Find the total cost for a group of 4 adults and 7 children. [2]

Q24. When the brakes of a car are fully applied the car will continue to travel some distance before it completely stops. This stopping distance, d , in metres is directly proportional to the square of the speed of the car, v , in kilometres per hour ($km\ h^{-1}$).

When a car is travelling at a speed of $50\ km\ h^{-1}$ it will travel $12.3\ m$ after the brakes are fully applied before it completely stops.

- (a) Determine an equation for d in terms of v .

The police can use this equation to estimate if cars are exceeding the speed limit.

A car is found to have travelled $33\ m$, after fully applying its brakes, before it completely stopped.

- (b) Use your equation from part (a) to estimate the speed at which this car was travelling before the brakes were applied.
- (c) After the brakes have been fully applied, identify one other variable besides speed that could affect stopping distance.

Q25. The height of a baseball after it is hit by a bat is modelled by the function

$$h(t) = -4.8t^2 + 21t + 1.2$$

where $h(t)$ is the height in metres above the ground and t is the time in seconds after the ball was hit.

- (a) Write down the height of the ball above the ground at the instant it is hit by the bat.
- (b) Find the value of t when the ball hits the ground.
- (c) State an appropriate domain for t in this model.

Q26.

In the realm of Artificial Intelligence (AI), machine learning models are trained using data sets. The time taken to train a machine learning model is directly proportional to the size of the training data set. This is shown by the following:

$$t \propto x$$

Where t is training time in hours, and x is the size of the training data set in gigabytes.

It is known that a certain machine learning model takes 15 hours to train using a data set of size 100 gigabytes.

- (a) Find an equation for t in terms of x . [2]
- (b) If a dataset of 250 gigabytes is used, determine the expected training time. [2]

The larger the data set that is used for training, the more effective the machine learning model is deemed to be.

A new and improved training model has been developed, so that a machine learning model can be trained in the same amount of time as the previous model, using larger data sets. The equation for the new model is $t = 0.1x$.

- (c) Given a training time of 20 hours, determine the difference in the size of the data sets between the first and second machine learning models. [3]

Q27.

Jasper owns a business and has 7 employees. He can assign any number of employees to a task. The time the task takes varies inversely with the number of employees who are assigned to it. This is shown by the following:

$$t \propto \frac{1}{x}$$

Where t is time a task takes in hours, and x is the number of employees assigned to the task.

If one employee is assigned to a task, it will take 8 hours to complete the task.

- (a) Find an equation for t in terms of x . [2]

Jasper has a task that must be completed within three hours.

- (b) (i) Determine the minimum number of employees that should be assigned to the task to ensure it is completed within three hours.
(ii) State the total amount of time the task will take. Give your answer to the nearest minute. [3]

If a task does not take a whole number of hours, Jasper must round $\lceil \text{time} \rceil$ to the nearest whole hour to pay his employees.

- (c) Determine the optimal number of employees Jasper should assign to ensure the task is complete within three hours, while minimising the number of hours of work he needs to pay for. [3]