Mathematics
Standard Level
Paper 1

Name

Date: ______________________

1 hour 30 minutes

SOLUTION KEY

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[80 marks]**.

worked solutions: 12 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A (35 marks)

Answer **all** questions in the boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The graph of a parabola has x-intercepts at $(-3,0)$ and $(1,0)$.

(a) Determine the equation of the axis of symmetry of the parabola. [2]

(b) An equation for the parabola can be written in the form $y = ax^2 + bx + c$. Given that the parabola passes through $(0,-6)$, find the value of a, the value of b, and the value of c. [4]

(a) The axis of symmetry is a vertical line. The points $(-3,0)$ and $(1,0)$ are equidistant from the axis of symmetry.

$$\frac{-3+1}{2} = -1$$

Thus, the axis of symmetry has the $x = -1$

(b) $x+3$ and $x-1$ are factors of the quadratic expression for the parabola

Hence, $y = a(x+3)(x-1) = a(x^2+2x-3)$

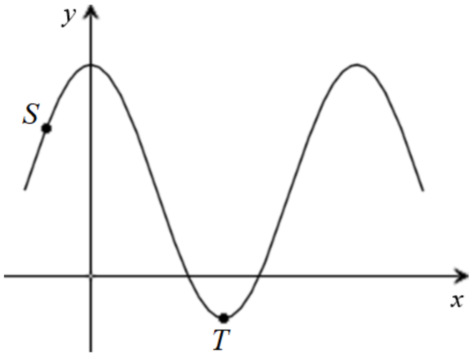
$(0,-6)$: $y = a(x^2+2x-3) = a(-3) = -6 \quad \Rightarrow \quad a = 2$

Substituting: $y = 2(x^2+2x-3) = 2x^2+4x-6$

Thus, $a = 2$, $b = 4$, $c = -6$

2. [Maximum mark: 6]

The diagram below shows a curve with equation $y = 2 + k\cos x$, defined for $-\frac{\pi}{2} \le x \le \frac{5\pi}{2}$.



The point S lies on the curve and has coordinates $\left(-\frac{\pi}{3}, \frac{7}{2}\right)$. The point T with coordinates (a, b) is the minimum point.

(a) Show that $k = 3$. [2]

(b) Hence, find the value of a and the value of b. [4]

(a) substitute in $\left(-\frac{\pi}{3}, \frac{7}{2}\right)$

$\frac{7}{2} = 2 + k\cos\left(-\frac{\pi}{3}\right)$

$\frac{7}{2} = 2 + k\left(\frac{1}{2}\right)$

$\frac{1}{2}k = \frac{3}{2} \Rightarrow k = 3$ Q.E.D.

(b) $y = 2 + 3\cos x \rightarrow \frac{dy}{dx} = -3\sin x = 0$

$\sin x = 0$ find first solution such that $x > 0$

$x = \pi \rightarrow a = \pi$

substituting: $b = 2 + 3\cos(\pi) = 2 + 3(-1) = -1$

thus, $a = \pi$, $b = -1$

3. [Maximum mark: 6]

A geometric series has a positive common ratio r. The series has a sum to infinity of 9 and the sum of the first two terms is 5. Find the first three terms of the series.

$S_\infty = \frac{u_1}{1-r} = 9 \Rightarrow u_1 = 9 - 9r$

$u_1 + u_1 r = 5$ substituting gives

$9 - 9r + (9-9r)r = 5$

$9 - 9r + 9r - 9r^2 = 5$

$9r^2 = 4$

$r^2 = 4/9$

$r = 2/3 \quad (r > 0)$

$u_1 = 9 - 9(2/3) = 9 - 6 = 3$

$u_2 = 3(2/3) = 2$

$u_3 = 2(2/3) = 4/3$

first three terms of the series: $3, 2, 4/3$

4. [Maximum mark: 6]

Find the equation of the line that is **normal** to the curve $y = 3x + e^{-x}$ at the point where $x = 0$.

$$\frac{dy}{dx} = 3 - e^{-x}$$

At $x = 0$, $\frac{dy}{dx} = 3 - e^{0} = 3 - 1 = 2$

So, gradient of normal to the curve at $x = 0$ is $m = -\frac{1}{2}$

At $x = 0$, $y = 3(0) + e^{0} = 1$

Substituting into point-gradient form for a straight line:

$$y - 1 = -\frac{1}{2}(x - 0) \quad \Rightarrow \quad y = -\frac{1}{2}x + 1$$

Thus, $y = -\frac{1}{2}x + 1$ is the equation of the normal to the curve at the point $(0,1)$.

5. [Maximum mark: 7]

Solve for x in each of the following equations:

(a) $\log_2\left(5x^2 - x - 2\right) = 2 + 2\log_2 x$. [3]

(b) $3^{x+1} = 2^{2-x}$. Express the answer in the form $\frac{\ln a}{\ln b}$, $a, b \in \mathbb{Q}$. [4]

(a) $\log_2(5x^2 - x - 2) = \log_2 4 + \log_2 x^2$

$\log_2(5x^2 - x - 2) = \log_2 4x^2$

thus, $5x^2 - x - 2 = 4x^2 \Rightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \rightarrow x = 2$ or ~~$x = -1$~~ not allowed

(b) $3^{x+1} = 2^{2-x}$

$\ln\left(3^{x+1}\right) = \ln\left(2^{2-x}\right)$

$(x+1)\ln 3 = (2-x)\ln 2$

$x\ln 3 + \ln 3 = 2\ln 2 - x\ln 2$

$x\ln 3 + x\ln 2 = \ln 2^2 - \ln 3$

$x(\ln 3 + \ln 2) = \ln 4 - \ln 3$

$x\ln 6 = \ln \frac{4}{3}$

$x = \frac{\ln \frac{4}{3}}{\ln 6}$

6. [Maximum mark: 6]

The coefficients of x^2 in the expansions $(1+x)^{2n}$ and $(1+15x^2)^n$ are equal. Given that n is a positive integer, find the value of n.

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

expressions for x^2 terms in each expansion

$(1+x)^{2n}$: x^2 term $\binom{2n}{2}(1)^{2n-2}(x)^2 = \binom{2n}{2}x^2$

$(1+15x^2)^n$: x^2 term $\binom{n}{1}(1)^{n-1}(15x^2)^1 = \binom{n}{1}15x^2$

equating the coefficients

$$\binom{2n}{2} = 15\binom{n}{1}$$

$$\frac{(2n)!}{2(2n-2)!} = 15n$$

$$2n(2n-1) = 30n$$

$$4n^2 = 32n$$

$$n = 8$$

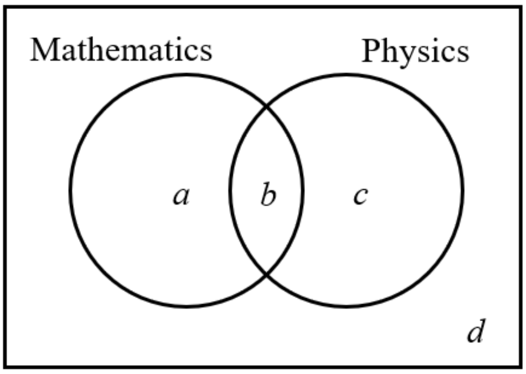
Do **not** write solutions on this page.

Section B (45 marks)

Answer **all** the questions on the answer sheets provided. Please start each question on a new page.

7. [Maximum mark: 14]

In a class of 16 students, 12 study Mathematics, 6 study Physics, and 2 study neither Mathematics or Physics. The class is represented by the diagram below.



(a) (i) Calculate the value of a, the value of b, the value of c and the value of d. [5]

(ii) A student is randomly selected from the class. Given that the student studies Physics, show that the probability that the student also studies Mathematics is $\frac{2}{3}$. [2]

(iii) Two students are randomly selected from the class. Find the probability that the first student chosen studies only Mathematics and the second student chosen studies only Physics. [4]

(b) Two students are randomly selected from the class and moved to a different class. Find the probability that the remaining class of 14 students has no students that studies only Physics. [3]

Do **not** write solutions on this page.

8. [Maximum mark: 19]

The velocity v, in $\mathrm{m\,s^{-1}}$, of an object moving along a straight line at time t seconds is given by $v(t) = 2t^2 - 4t - 6\ \mathrm{m\,s^{-1}},\ 0 \le t \le 4$. At $t = 0$ seconds, the displacement of the object is $s = 0$ m.

(a) (i) Determine the value of t at which the object reaches its minimum velocity.

(ii) Show that the displacement of the object at this value of t is $-\frac{22}{3}$ m. [7]

(b) (i) Sketch a graph of $v(t)$, clearly labelling all axis intercepts, maxima and minima.

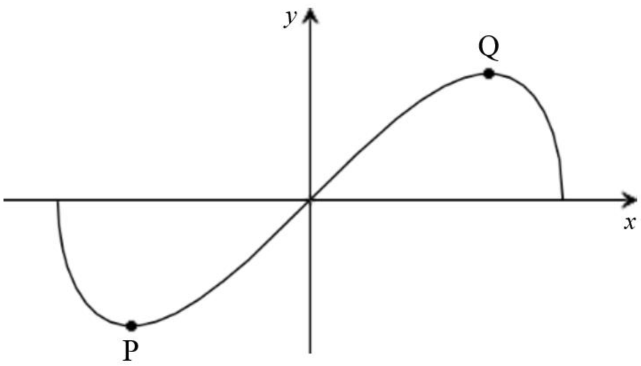
(ii) Write down the interval of time during which the object is moving to the right.

(iii) Write down the interval of time during which the object is moving to the left. [8]

(c) Find the distance travelled by the object from $t = 0$ to $t = 2$ seconds. [4]

9. [Maximum mark: 10]

The diagram shows the graph of the function defined by $f(x) = x\sqrt{1-x^2},\ -1 \le x \le 1$.



The function has a minimum at the point P and a maximum at point Q.

(a) Show that $f'(x) = \dfrac{1-2x^2}{\sqrt{1-x^2}}$. [4]

(b) Find the coordinates of P, and the coordinates of Q. [4]

(c) Given that the function g is defined as $g(x) = 2f(x-3)$, determine the range of g. [2]

■ Worked solutions for questions 7, 8 & 9 ■

7. (a) (i) 2 students study neither Mathematics or Physics, so $d = 2$

hence $a + b + c = 14$

$$\left.\begin{aligned} a + b &= 12 \\ b + c &= 6 \end{aligned}\right\} a - c = 6 \;\Rightarrow\; a = c + 6$$

substituting: $c + 6 + b + c = 14 \;\Rightarrow\; b + 2c = 8$

$$\left.\begin{aligned} b + 2c &= 8 \\ b + c &= 6 \end{aligned}\right\} c = 2 \;\Rightarrow\; b = 4,\; a = 8$$

(ii) Using conditional probability formula:

$$\mathrm{P}(\text{Math} \mid \text{Physics}) = \frac{\mathrm{P}(\text{Math} \cap \text{Physics})}{\mathrm{P}(\text{Physics})} = \frac{b}{b + c} = \frac{4}{4 + 2} = \frac{2}{3}$$ ***Q.E.D.***

(iii) For the 1st student, there are 16 students in the class and 8 students study only Math.

hence, $\mathrm{P}(\text{1st Math only}) = \frac{8}{16} = \frac{1}{2}$

For the 2nd student, there are 15 students in the class and 2 students study only Physics.

hence, $\mathrm{P}(\text{2nd Physics only}) = \frac{2}{15}$

Therefore, $\mathrm{P}(\text{1st Math only} \cap \text{2nd Physics only}) = \frac{1}{2} \cdot \frac{2}{15} = \frac{1}{15}$

(b) The number of students in the initial class is 16, and 2 of them are studying only Physics.
In order for the reduced class of 14 to have no students studying only Physics then both of the randomly selected students must be studying only Physics.

The probability that the first selected student studies only Physics $= \frac{2}{16}$

The probability that the second selected student studies only Physics $= \frac{1}{15}$

Thus, the probability that the remaining class of 14 students has no students

that study only Physics $= \frac{2}{16} \cdot \frac{1}{15} = \frac{1}{8} \cdot \frac{1}{15} = \frac{1}{120}$

8. (a) (i) min. velocity occurs at $t = t_{\min}$ where $v'(t_{\min}) = 0$ and $v''(t_{\min}) > 0$ (graph of v concave up)

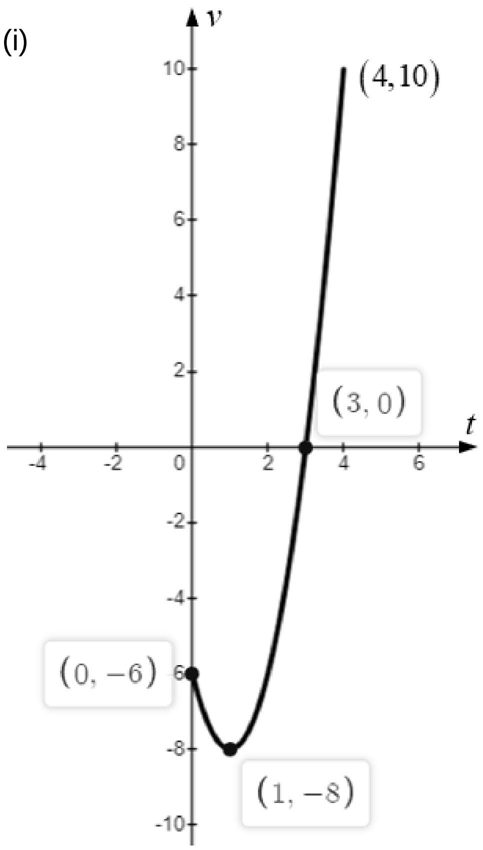
$v'(t) = 4t - 4 = 0 \Rightarrow t = 1$

$v''(t) = 4 > 0$ for all t Thus, minimum velocity occurs when $t = 1$ second

(ii) displacement: $s(t) = \int_0^t (2t^2 - 4t - 6)\,\mathrm{d}t = \left.\frac{2}{3}t^3 - 2t^2 - 6t\right]_0^t$

At $t = 1$: $s = \frac{2}{3}(1)^3 - 2(1)^2 - 6(1) = \frac{2}{3} - 2 - 6 = \frac{2}{3} - \frac{6}{3} - \frac{18}{3} = -\frac{22}{3}\,\text{m}$ **Q.E.D.**

(b) (i)



At $v = 0$, $2t^2 - 4t - 6 = 0 \Rightarrow t^2 - 2t - 3 = 0$

$\Rightarrow (t+1)(t-3) = 0 \Rightarrow t = 3$ since $0 \le t \le 4$

Therefore, x-intercept at $(3, 0)$

At $t = 0$, $v = 2(0)^2 - 4(0) - 6 = -6$

Therefore, y-intercept at $(0, -6)$

At $t = 1$, $v = 2(1)^2 - 4(1) - 6 = 2 - 4 - 6 = -8$

Therefore, minimum point at $(1, -8)$

Maximum occurs when $x = 4$

$v(4) = 2(4)^2 - 4(4) - 6 = 0 \Rightarrow v(4) = 10$

Therefore, maximum point at $(4, 10)$

(b) (ii) The object moves to the right when $v(t) > 0$; hence moving right during interval $3 < t \le 4$

(iii) The object moves to the left when $v(t) < 0$; hence moving left during interval $0 \le t < 3$

(c) The object is moving left continually during $0 \le t \le 2$, so take absolute value of definite integral for computing displacement to find to distance travelled from $t = 0$ to $t = 2$

distance $= \left|\int_0^2 (2t^2 - 4t - 6)\,\mathrm{d}t\right| = \left|\left[\frac{2}{3}t^3 - 2t^2 - 6t\right]_0^2\right|$

$= \left|\left[\left(\frac{2}{3}(2)^3 - 2(2)^2 - 6(2)\right) - 0\right]\right| = \left|\frac{16}{3} - 8 - 12\right|$

Thus, distance $= \frac{44}{3} \approx 14.7$ meters

9. (a) $f(x) = x\left(1-x^2\right)^{\frac{1}{2}}$

$$f'(x) = \left(1-x^2\right)^{\frac{1}{2}} + x\left[\tfrac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\right]$$

$$= \left(1-x^2\right)^{\frac{1}{2}} - x^2\left(1-x^2\right)^{-\frac{1}{2}}$$

$$= \left(1-x^2\right)^{-\frac{1}{2}}\left[\left(1-x^2\right)^{1} - x^2\right]$$

$$= \left(1-x^2\right)^{-\frac{1}{2}}\left(1-2x^2\right)$$

$$f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}} \quad \underline{\text{Q.E.D.}}$$

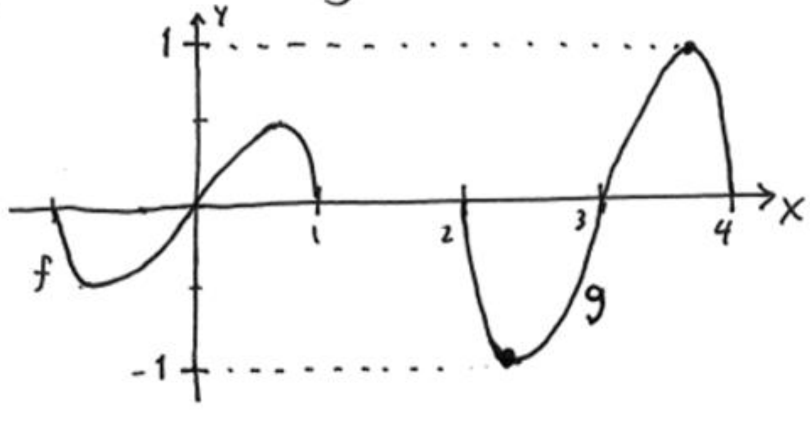
(b) $f'(x) = \dfrac{1-2x^2}{\sqrt{1-x^2}} = 0 \Rightarrow 1-2x^2 = 0 \rightarrow x^2 = \frac{1}{2} \rightarrow x = \pm\frac{\sqrt{2}}{2}$

$$f\left(\tfrac{\sqrt{2}}{2}\right) = \tfrac{\sqrt{2}}{2}\sqrt{1-\left(\tfrac{\sqrt{2}}{2}\right)^2} = \tfrac{\sqrt{2}}{2}\sqrt{\tfrac{1}{2}} = \tfrac{\sqrt{2}}{2}\cdot\tfrac{\sqrt{2}}{2} = \tfrac{1}{2}$$

$$f\left(-\tfrac{\sqrt{2}}{2}\right) = -\tfrac{\sqrt{2}}{2}\sqrt{1-\left(-\tfrac{\sqrt{2}}{2}\right)^2} = -\tfrac{\sqrt{2}}{2}\cdot\tfrac{\sqrt{2}}{2} = -\tfrac{1}{2}$$

thus, $P\left(-\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$ and $Q\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

(c) graph of g can be found by translating graph of f 3 units to the right and stretching it vertically by a factor of 2



range of g: $-1 \le y \le 1$