

## SL Paper 1 Mock B 2021 – WORKED SOLUTIONS v1

## Section A

1. Two points  $P(-5, 4)$  and  $Q(11, -8)$  are on the line  $L$ , so  $L$  has gradient

$$m = \frac{4 - (-8)}{-5 - 11} = \frac{12}{-16} = -\frac{3}{4}$$

The gradient of the line perpendicular to  $L$  is therefore  $m_{\perp} = \frac{4}{3}$

The midpoint of  $[PQ]$  is  $\left(\frac{-5+11}{2}, \frac{4+(-8)}{2}\right) \Rightarrow \left(\frac{6}{2}, \frac{-4}{2}\right) \Rightarrow (3, -2)$

Solving for the equation of the line perpendicular to  $L$ :

$$y - (-2) = \frac{4}{3}(x - 3) \Rightarrow y + 2 = \frac{4}{3}x - 4 \Rightarrow y = \frac{4}{3}x - 6$$

Thus, the line perpendicular to  $L$  that passes through  $(3, -2)$  has equation  $y = \frac{4}{3}x - 6$

2. (a)  $h(x) = (f \circ g)(x) = f(g(x))$

$$f(g(x)) = f(2x - 3)$$

$$= \frac{1}{2(2x - 3) + 1}$$

$$= \frac{1}{4x - 6 + 1}$$

$$f(g(x)) = \frac{1}{4x - 5}$$

$$\text{thus, } h(x) = \frac{1}{4x - 5}$$

- (b)  $y = \frac{1}{4x - 5}$ ; solve for  $x$

$$\frac{1}{y} = 4x - 5 \Rightarrow 4x = \frac{1}{y} + 5 \Rightarrow x = \frac{1}{4y} + \frac{5}{4}$$

$$\text{thus, } h^{-1}(x) = \frac{1}{4x} + \frac{5}{4} \quad \text{or, equivalently, } h^{-1}(x) = \frac{5x + 1}{4x}$$

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3. (a)  $g'(x) = (x-4)^3$ ; find  $g''(x)$  using the chain rule

let  $u = x-4$  and let  $y = u^3$

$$\frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = 1$$

$$g''(x) = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 1$$

undo substitution:

$$g''(x) = 3(x-4)^2$$

(b)  $g''(4) = 3(4-4)^2 = 0$

(c)  $g''(x) \geq 0$  for all values of  $x$ , meaning that the graph of  $g(x)$  is always concave up. An inflexion point requires a change in sign of the value of  $g''(x)$ , i.e. it requires the graph of  $g(x)$  to go from concave up to concave down, or vice versa. As  $g(x)$  is always concave up, neither point  $A$  nor any other point on  $g(x)$  is an inflexion point.

4. (a)  $\log_3 20 = \log_3 (4 \cdot 5) = \log_3 (2^2) + \log_3 5 = 2\log_3 2 + \log_3 5 = 2x + y$

thus,  $\log_3 20 = 2x + y$

(b)  $\log_3 \left(7\frac{13}{16}\right) = \log_3 \left(\frac{7 \cdot 16 + 13}{16}\right) = \log_3 \left(\frac{112 + 13}{16}\right) = \log_3 \left(\frac{125}{16}\right)$

$$\log_3 \left(\frac{125}{16}\right) = \log_3 125 - \log_3 16 = \log_3 (5^3) - \log_3 (2^4) = 3\log_3 5 - 4\log_3 2 = 3y - 4x$$

thus,  $\log_3 \left(7\frac{13}{16}\right) = 3y - 4x$

(c)  $\log_5 8 = \frac{\log_3 8}{\log_3 5} = \frac{\log_3 (2^3)}{\log_3 5} = \frac{3\log_3 2}{\log_3 5} = \frac{3x}{y}$

thus,  $\log_5 8 = \frac{3x}{y}$

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5. (a)  $1 + \ln x + (\ln x)^2 + \dots$  is a geometric sequence with  $u_1 = 1$  and  $r = \ln x$

a geometric sequence converges when  $|r| < 1$

$$|\ln x| < 1 \Rightarrow -1 < \ln x < 1$$

$$\ln x < 1 \Rightarrow x < e$$

$$\ln x > -1 \Rightarrow x > e^{-1}$$

thus, the sequence converges when  $\frac{1}{e} < x < e$

(b) the sum of an infinite geometric sequence is  $S_\infty = \frac{u_1}{1-r}$

for this sequence,  $u_1 = 1$  and  $r = \ln x$

$$S_\infty = \frac{1}{1-\ln x}; \text{ we want to find the value of } x \text{ such that } S_\infty = 2$$

$$\frac{1}{1-\ln x} = 2 \Rightarrow 1 - \ln x = \frac{1}{2} \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{\frac{1}{2}} = \sqrt{e}$$

thus, the sequence converges to 2 when  $x = \sqrt{e}$

6.  $8 \sin x \cos x = \sqrt{12}$ ; simplify LHS using trigonometric identities

$$\text{LHS} = 8 \sin x \cos x = 4(2 \sin x \cos x) = 4 \sin 2x$$

simplify RHS:

$$\text{RHS} = \sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$$

$$4 \sin 2x = 2\sqrt{3} \Rightarrow \sin 2x = \frac{\sqrt{3}}{2} \Rightarrow 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

we are told to solve the equation for  $0 \leq x \leq \frac{\pi}{2}$ , i.e.  $0 \leq 2x \leq \pi$

$$\text{therefore, } 2x = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\text{hence, } x = \frac{\pi}{6}, \frac{\pi}{3}$$

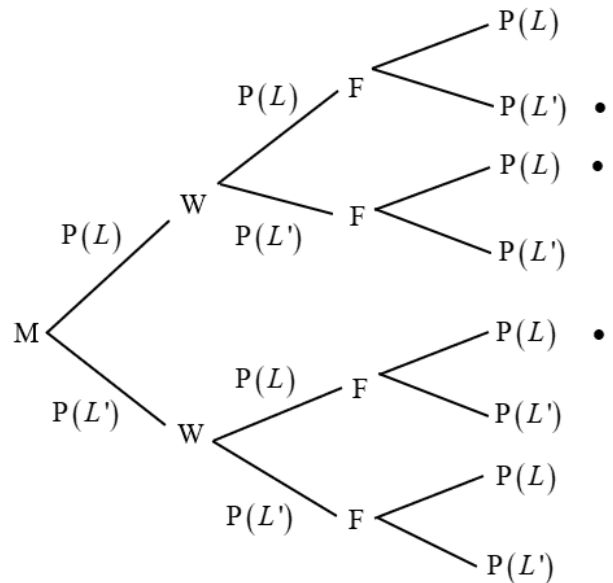
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**Section B**

7. (a)  $\frac{1}{8} + p = 1 \Rightarrow p = \frac{7}{8}$
- (b)  $P(T \cap L) = \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{12}$
- (c)  $P(L) = \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$
- (d)  $P(T | L) = \frac{P(T \cap L)}{P(L)} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$
- (e) on any given school day, the

probability that Sophie will be late is  $P(L) = \frac{1}{6}$

construct a tree diagram of the three days that Sophie goes to school, with M for Monday, W for Wednesday and F for Friday. Each day, there is a probability of being late,  $P(L)$ , and a probability of not being late,  $P(L')$ .



Out of the 8 possible outcomes, 3 are such that Sophie is late for school exactly twice in the week. These are labelled in the tree diagram above by a black circle. Therefore, the probability that Sophie is late for school exactly twice next week is the probability of any of these 3 outcomes occurring.

Each of the 3 outcomes have an equal probability of occurring, which is  $P(L) \cdot P(L) \cdot P(L')$

So, the probability of Sophie being late exactly twice next week is  $3(P(L) \cdot P(L) \cdot P(L'))$

Substituting  $P(L) = \frac{1}{6}$  and  $P(L') = \frac{5}{6}$ :

$$3 \left( \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \right) = \frac{1}{2} \cdot \frac{5}{36} = \frac{5}{72}$$

Thus, the probability that Sophie will be late exactly twice next week is  $\frac{5}{72}$ .

**Alternatively**, the answer can be found directly as a binomial distribution where  $X \sim B\left(3, \frac{1}{6}\right)$ .

$$\text{Then, } P(X = 2) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right) = \frac{5}{72} \approx 0.0694$$

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8. (a)  $f(x) = 2x^2 + 4x - 5 = 2(x^2 + 2x) - 5$ ; completing the square

$$f(x) = 2(x^2 + 2x + 1) - 5 - 2$$

$$f(x) = 2(x+1)^2 - 7$$

(b) (i)  $(-1, -7)$

(ii)  $x = -1$

(iii)  $(0, -5)$

(iv)  $x$ -intercepts exist where  $f(x) = 0$

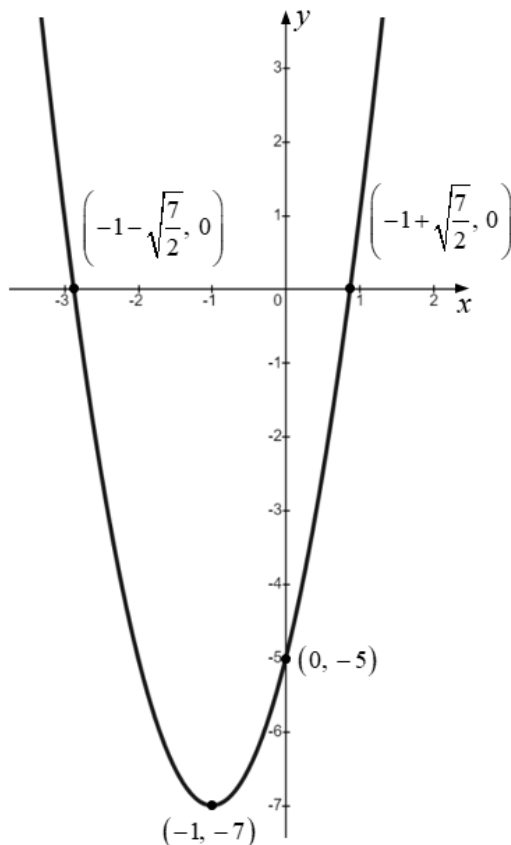
$$2(x+1)^2 - 7 = 0 \Rightarrow 2(x+1)^2 = 7$$

$$(x+1)^2 = \frac{7}{2} \Rightarrow x+1 = \pm\sqrt{\frac{7}{2}}$$

$$x = -1 \pm \sqrt{\frac{7}{2}}$$

thus,  $x$ -intercepts are  $x = -1 + \sqrt{\frac{7}{2}}$  and  $x = -1 - \sqrt{\frac{7}{2}}$

(c)



$$\sqrt{\frac{7}{2}} < \sqrt{\frac{8}{2}} = \sqrt{4} = 2$$

thus,  $\sqrt{\frac{7}{2}}$  is slightly less than 2

hence, it follows that  $x = -1 + \sqrt{\frac{7}{2}}$  is slightly less than 1, and

$x = -1 - \sqrt{\frac{7}{2}}$  is slightly greater than -3

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9. (a)  $g(x) = \frac{3x}{x^2+7}$ ; find  $g'(x)$  using quotient rule

$$\text{let } u = 3x \Rightarrow \frac{du}{dx} = 3$$

$$\text{let } v = x^2 + 7 \Rightarrow \frac{dv}{dx} = 2x$$

$$g'(x) = \frac{3 \cdot (x^2 + 7) - 3x \cdot 2x}{(x^2 + 7)^2}$$

$$= \frac{3x^2 + 21 - 6x^2}{(x^2 + 7)^2}$$

$$g'(x) = \frac{21 - 3x^2}{(x^2 + 7)^2} \quad \text{Q.E.D.}$$

(b)  $\int \frac{3x}{x^2+7} dx$ ; using integration by substitution

$$\text{let } u = x^2 + 7 \Rightarrow \frac{du}{dx} = 2x$$

$$\int \frac{3x}{x^2+7} dx = \frac{3}{2} \int \frac{1}{u} du$$

$$= \frac{3}{2} \ln u + C; \text{ undo substitution}$$

$$\int \frac{3x}{x^2+7} dx = \frac{3}{2} \ln(x^2 + 7) + C$$

(c)  $\text{area} = \int_{\sqrt{7}}^a g(x) dx$

$$= \int_{\sqrt{7}}^a \frac{3x}{x^2+7} dx$$

$$\text{from (b), } \int \frac{3x}{x^2+7} dx = \frac{3}{2} \ln(x^2 + 7) + C$$

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$$\begin{aligned}\text{thus, area} &= \left[ \frac{3}{2} \ln(x^2 + 7) \right]_{\sqrt{7}}^a \\ &= \frac{3}{2} \ln(a^2 + 7) - \frac{3}{2} \ln\left((\sqrt{7})^2 + 7\right)\end{aligned}$$

$$= \frac{3}{2} \ln(a^2 + 7) - \frac{3}{2} \ln 14$$

$$= \frac{3}{2} \ln\left(\frac{a^2 + 7}{14}\right)$$

we are given that the area of the region is  $\ln 8$

$$\text{hence, } \frac{3}{2} \ln\left(\frac{a^2 + 7}{14}\right) = \ln 8$$

$$\ln\left(\frac{a^2 + 7}{14}\right) = \frac{2}{3} \ln 8 = \ln\left(8^{\frac{2}{3}}\right) = \ln\left((2^3)^{\frac{2}{3}}\right) = \ln(2^2)$$

$$\ln\left(\frac{a^2 + 7}{14}\right) = \ln 4$$

$$\frac{a^2 + 7}{14} = 4 \Rightarrow a^2 + 7 = 56$$

$$a^2 = 56 - 7 = 49 \Rightarrow a = \pm 7$$

$$a > \sqrt{7}, \text{ therefore } a = 7$$